1. Sea $f(x,y) = 3x^3y - 2x^2y^2 + y^3$. Determinar $f_x(1,-2)$ y $f_y(1,-2)$.
   Solución:

   Como $f_x(x,y) = 9x^2y - 4xy^2$, luego:

   $$f_x(1,-2) = -34$$

   Como $f_y(x,y) = 3x^3 - 4x^2y + 3y^2$, luego:

   $$f_y(1,-2) = 23$$

2. Sea $z = f(x,y) = \ln(x^2 + y)$.

   a) Determinar $f_x(1,2)$ y $f_y(1,2)$.

   b) Determinar las segundas derivadas parciales: f_{xx}, f_{xy}, etc.

   Solución:

   a) $f_x(x,y) = \dfrac{\partial f}{\partial x}(x,y) = \dfrac{2x}{x^2+y}$. Luego, $f_x(1,2) = \dfrac{2}{3}$

   $f_y(x,y) = \dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{x^2+y}$. Luego, $f_y(1,2) = \dfrac{1}{3}$

   b) $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\left(\dfrac{2x}{x^2+y}\right) = \dfrac{2(y-x^2)}{(x^2+y)^2}$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{2x}{x^2+y}\right) = -\dfrac{2x)}{(x^2+y)^2}$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\left(\dfrac{1}{x^2+y}\right) = -\dfrac{2x)}{(x^2+y)^2}$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{1}{x^2+y}\right) = -\dfrac{1)}{(x^2+y)^2}$

3. Sea $u = f(x,y) = e^x \sin(y)$. Probar que: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

   Solución:

   $$\frac{\partial u}{\partial x} = e^x \sin(y) \qquad \frac{\partial u}{\partial y} = e^x \cos(y)$$

   $$\frac{\partial^2 u}{\partial x^2} = e^x \sin(y) \qquad \frac{\partial^u}{\partial y^2} = -e^x \sin(y)$$

   Luego, $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = e^x \sin(y) - e^x \sin(y) = 0$

4. $z = f(x,y) = x\,e^{y^2} + y \ln x$. Verificar que $f_{xy} = f_{yx}$

   Solución:

   Como:

$$f_x(x,y) = e^{y^2} + \frac{y}{x} \qquad \text{y} \qquad f_y(x,y) = 2xy\,e^{y^2} + \ln x$$

   Luego:

$$f_{xy}(x,y) = 2y\,e^{y^2} + \frac{1}{x} \qquad \text{y} \qquad f_{yx}(x,y) = 2y\,e^{y^2} + \frac{1}{x}$$

   Por lo tanto, $f_{xy} = f_{yx}$

5. Sea $u = \dfrac{x^2y^2}{x+y}$. Demostrar que: $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$.

   Solución:

$$\frac{\partial u}{\partial x} = \frac{2xy^2(x+y) - x^2y^2\cdot 1}{(x+y)^2} = \frac{2x^2y^2 - 2xy^3 - x^2y^2}{(x+y)^2} = \frac{x^2y^2 + 2xy^3}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{x^2y^2 + 2x^3y}{(x+y)^2}$$

   Sustituyendo en la ecuación, se obtiene:

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= x\frac{x^2y^2 + 2xy^3}{(x+y)^2} + y\frac{x^2y^2 + 2x^3y}{(x+y)^2} \\ &= \frac{x\cdot xy^2(x+2y) + y\cdot x^2y(2x+y)}{(x+y)^2} = \frac{x^2y^2(3x+3y)}{(x+y)^2} \\ &= \frac{3x^2y^2(x+y)}{(x+y)^2} = 3\cdot\frac{x^2y^2}{x+y} = 3u \end{aligned}$$

6. Determinar $a \in \mathbb{R}$, de modo que la función: $f(x,t) = \cos(2cx + act)$, con $c \neq 0$, satisfaga la ecuación:

$$\frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2}$$

   Solución:

$$\frac{\partial f}{\partial t} = -ac\sin(2x + act) \implies \frac{\partial^2 f}{\partial t^2} = -a^2c^2\cos(2x + act)$$

$$\frac{\partial f}{\partial x} = -2c\sin(2cx + act) \implies \frac{\partial^2 f}{\partial x^2} = -4c^2\cos(2x + act)$$

   Luego:

$$\begin{aligned} \frac{\partial^2 f}{\partial t^2} = c^2\frac{\partial^2 f}{\partial x^2} &\implies -a^2c^2\cos(2x + act) = -4c^2\cos(2x + act) \\ &\implies a^2 = 4 \end{aligned}$$

   Luego, los valores de a que satisfacen la ecuación propuesta son 2 y-2.

7. Calcular la pendiente de la recta tangente a la curva de interseccioón de la superficie:

$$36x^2 - 9y^2 + 4z^2 + 36 = 0$$

con el plano $x = 1$, en el punto $(1, \sqrt{12}, -3)$.

Solución:

Notar que el punto $(1, \sqrt{2}, -1)$ pertenece a la superficie, ya que:

$$36 \cdot 1^2 - 9 \cdot (\sqrt{12})^2 + 4 \cdot (-3)^2 + 36 = 36 - 108 + 36 + 36 = 0$$

Se tiene que, la pendiente de la recta tangente a la curva de interseccioón de la superficie $36x^2 - 9y^2 + 4z^2 + 36 = 0$ con el plano $x = 1$, en el punto $(1, \sqrt{12}, -3)$ es:

$$z_y \text{ evaluado en el punto } (1, \sqrt{12}, -3)$$

Se calculará en primer lugar: z_y.

$$-18y + 8z \cdot z_y = 0 \Longrightarrow z_y = \frac{9y}{4z}$$

Luego, la pendiente de la recta tangente a la curva de interseccioón de la superficie $36x^2 - 9y^2 + 4z^2 + 36 = 0$ con el plano $x = 1$, en el punto $(1, \sqrt{12}, -3)$ es:

$$-\frac{3\sqrt{12}}{4}$$

8. Sea $u = (ax^2 + by^2 + cz^2)^3$. Demostrar que:

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial^3 u}{\partial y \partial x^2}$$

Solución:

Sea $v = ax^2 + by^2 + cz^2$. Luego $u = v^3$ y $\frac{\partial v}{\partial x} = 2ax$, $\frac{\partial v}{\partial y} = 2by$, $\frac{\partial v}{\partial z} = 2cz$. q

$$\begin{aligned}
\frac{\partial^3 u}{\partial x^2 \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} v^3\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(3v^2 \frac{\partial v}{\partial y}\right)\right) = 3\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(v^2 \cdot 2by\right)\right) \\
&= 6by\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(v^2\right)\right) = 6by\frac{\partial}{\partial x}\left(2v\frac{\partial v}{\partial x}\right) = 12by\frac{\partial}{\partial x}\left(2axv\right) \\
&= 24aby\frac{\partial}{\partial x}(xv) = 24aby\left(v + x \cdot \frac{\partial}{\partial x} v\right) = 24aby(v + 2ax^2) \\
&= 24aby(3ax^2 + by^2 + cz^2).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 u}{\partial x \partial y \partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}v^3\right)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(3v^2\frac{\partial v}{\partial x}\right)\right) = 3\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(v^2 \cdot 2ax\right)\right) \\
&= 6a\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial y}v^2\right) = 6a\frac{\partial}{\partial x}\left(x2v\frac{\partial v}{\partial y}\right) = 6a\frac{\partial}{\partial x}\left(2xv \cdot 2by\right) \\
&= 24aby\frac{\partial}{\partial x}\left(xv\right) = 24aby\left(v + x \cdot \frac{\partial v}{\partial x}\right) = 24aby(v + 2ax^2) \\
&= 24aby(3ax^2 + by^2 + cz^2).
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 u}{\partial y \partial x^2} &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}v^3\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(3v^2\frac{\partial v}{\partial x}\right)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(3v^2 \cdot 2ax\right)\right) \\
&= 6a\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}v^2 x\right) = 6a\frac{\partial}{\partial y}\left(v^2 + 2v\frac{\partial v}{\partial x} \cdot x\right) = 6a\frac{\partial}{\partial y}\left(v^2 + 2v \cdot 2ax \cdot x\right) \\
&= 6a\frac{\partial}{\partial y}\left(v^2 + 4avx^2\right) = 6a\left[\frac{\partial v^2}{\partial y} + 4ax^2 \cdot \frac{\partial v}{\partial y}\right] = 6a\left[2v \cdot \frac{\partial v}{\partial y} + 4ax^2 \cdot 2by\right] \\
&= 6a(2v \cdot 2by + 4ax^2 \cdot 2by) = 24aby(v + 2ax^2) = 24aby(3ax^2 + by^2 + cz^2).
\end{aligned}$$

Luego

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x} = \frac{\partial^3 u}{\partial y \partial x^2}.$$

9. Dada la siguiente función

$$u = u(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{si} \quad (x,y) \neq (0,0) \\ 0 & \text{si} \quad (x,y) = (0,0) \end{cases}$$

Calcular:

a) $u_x(x,y)$ y $u_y(x,y)$

b) $u_{xy}(0,0)$

c) $u_{yx}(0,0)$

Solución:

a) Para $(x,y) \neq (0,0)$:

$$u_x(x,y) = \frac{(y(x^2 - y^2) + 2x^2y)(x^2 + y^2) - 2x^2y(x^2 - y^2)}{(x^2 + y^2)^2} = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

para $(x,y) = (0,0)$:

$$u_x(0,0) = \lim_{t \to 0} \frac{u(t,0)}{t} = \lim_{t \to 0} = 0$$

Por lo tanto:

$$u_x(x,y) = \begin{cases} \dfrac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{si} \quad (x,y) \neq (0,0) \\ 0 & \text{si} \quad (x,y) = (0,0) \end{cases}$$

Anaálogamente, se obtiene que:

$$u_y(x,y) = \begin{cases} \dfrac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2} & \text{si} \quad (x,y) \neq (0,0) \\ 0 & \text{si} \quad (x,y) = (0,0) \end{cases}$$

b) $u_{xy}(0,0) = (u_x)_y(0,0) = \lim\limits_{t\to 0} \dfrac{u_x(0,t)-u_x(0,0)}{t} = \lim\limits_{t\to 0} = \dfrac{-t-0}{t} = -1$

c) $u_{yx}(0,0) = (u_y)_x(0,0) = \lim\limits_{t\to 0} \dfrac{u_y(t,0)-u_y(0,0)}{t} = \lim\limits_{t\to 0} = \dfrac{t-0}{t} = 1$